

# Dilation of states and processes in operational-probabilistic theories

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This paper provides a concise summary of the framework of operational-probabilistic theories, aimed at emphasizing the interaction between category-theoretic and probabilistic structures. Within this framework, we review an operational version of the GNS construction, expressed by the so-called purification principle [12], which under mild hypotheses leads to an operational version of Stinespring's theorem.

## 1 Introduction

Few theories in physics have been as successful and as surprising as Quantum Theory. Both successes and surprises come from a simple mathematical framework of virtually universal applicability, which blends physics and information theory in peculiar and often puzzling way. At the heart of this framework is a set of mathematical theorems, known as *dilation theorems* [33], which allow one to reduce all possible states, evolutions, and measurements allowed by quantum mechanics to some privileged subsets. Specifically,

1. mixed states are reduced to pure states (by the GNS construction [26, 34], familiar to the quantum information community as *purification*)
2. general evolutions are reduced to reversible evolutions (by Stinespring's theorem [37])
3. general measurements are reduced to sharp measurements (by Naimark's [31] and Ozawa's [32] theorems).

For finite dimensional quantum systems and for the simplest examples of infinite dimensional systems (type 1 von Neumann algebras), the reductions 1-3 are achieved by introducing an auxiliary system (the *environment*), which is eventually discarded. This fact lends itself to an operational interpretation: The ignorance about the preparation of a system, the irreversibility of an evolution, and the unsharpness of a measurement can always be explained as resulting from the lack of control over some degree of freedom in the surrounding environment.

Dilation theorems are usually regarded as a consequence of the mathematical framework of Quantum Theory. They are heavily employed as technical tools by researchers in quantum information theory, to the extent that one can hardly find results that do not invoke any of them, at least in an implicit way. The dilation approach is so common in quantum information that it earned itself a nickname—the “Church of the Larger Hilbert Space”<sup>1</sup>. However, the operational content of the dilation theorems is independent

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<sup>1</sup>The expression is due to John Smolin, see e.g. the lecture notes [9].

of the quantum framework: Even forgetting about Hilbert spaces and operator algebras, one can still express the notions of pure/mixed state, reversible/irreversible evolution, and sharp/unsharp measurement in a general framework of operational-probabilistic theories [27, 8, 24, 5, 6, 12, 25, 13, 7, 28, 29]. In this broader framework the dilation of states, evolutions and measurements can be promoted to the rank of *axioms*, from which (a number of features of) the theory is derived ([12]) [13]. There are at least three good reasons to follow this route. First, given the amount of results that invoke dilation theorems in quantum information processing, turning these theorems into axioms seems to be a convenient way to restructure the landscape of quantum information and to facilitate the discovery of new protocols. Second, the dilation approach sheds light on the old question “Why the quantum?”, the question of finding a set of well-motivated axioms that single out quantum theory among all possible theories. This is the path followed by Ref. [13] where the finite-dimensional Hilbert space framework has been reconstructed from the purification of mixed states—the so-called *Purification Principle*. In the light of this result, Quantum Theory appears as the golden standard of theory where information-theoretic notions admit a description in terms of pure states and fundamentally reversible interactions [14]. Third, the approach of abstracting dilation theorems from the Hilbert space framework yielded deep insights in category theory, leading to the formulation of Selinger’s CPM construction [35] and to its axiomatization in terms of interaction with the environment [18, 22].

In the following we review the framework of operational-probabilistic theories and the basic results about the Purification Principle, providing operational versions of the GNS construction and of Stinespring’s theorem in the finite-dimensional setting. The goal of this presentation is to provide a concise and mathematically rigorous summary of the state of the art. Overall, its contribution does not consist in new results, but rather in the systematization of the existing ones, which are put here in a more compact form. Since this is meant to be a summary of results, we will not provide proofs, most of which can anyways be recovered from the original work [12].

## 2 The framework of operational-probabilistic theories

In this section we review the framework of operational-probabilistic theories [12]. The framework consists of two distinct conceptual ingredients: an operational structure, describing circuits that produce outcomes, and a probabilistic structure, which assigns probabilities to the outcomes in a consistent way. The operational structure is closely related to and partly inspired by the area of categorical quantum mechanics [1, 17, 2]. There are nevertheless a few relevant differences in the way classical outcomes are treated, which turn out to be important when introducing the probabilistic structure.

### 2.1 Operational structure

The operational structure summarizes all the possible circuits that can be constructed in a given physical theory. In general, the circuits can consist of non-deterministic gates, where evolution of the input branches into a number of alternative processes  $\{\mathcal{M}_x\}_{x \in X}$  labelled by a finite index set  $X$ .

#### 2.1.1 The category of transformations

For a given non-deterministic gate  $\{\mathcal{M}_x\}_{x \in X}$ , each process  $\mathcal{M}_x$  is regarded as a physical transformation, transforming an input system  $A$  into an output system  $B$ . The set of all physical transformations is required to be a (strict) symmetric monoidal category (SMC) [4], denoted by  $\text{Transf}$ , and the set of all

transformations of type  $A \rightarrow B$  is denoted as  $\text{Transf}(A \rightarrow B)$ . The transformations of type  $I$  to  $I$  are also referred to as *scalars*. The identity morphism on system  $A$  is called the *identity transformation* and is denoted by  $\mathcal{I}_A$ . The identity on the tensor unit  $I$  will be denoted by  $1$ . The sequential and parallel composition of transformations are represented in the graphical language of SMCs [36, 19], which consists of circuits like



### 2.1.2 The category of tests

A non-deterministic gate of type  $A \rightarrow B$  is an indexed list of transformations in  $\{\mathcal{M}_x\}_{x \in X} \subset \text{Transf}(A \rightarrow B)$ . The index set  $X$  is called the *outcome space* and represents the possible outcomes that can occur when the gate is used. We denote by *Outcomes* the set of all possible outcome spaces appearing in the theory. For reasons that will be immediately clear, we require that

1. Outcomes is closed under Cartesian product
2. Outcomes contains the singleton set  $\{\varepsilon\}$ , where  $\varepsilon$  is the empty word, so that  $\{\varepsilon\} \times X = X \times \{\varepsilon\} = X$  for every  $X \in \text{Outcomes}$ .

Following the terminology of Ref. [12], we refer to non-deterministic gates as *tests* and we denote the tests of type  $A \rightarrow B$  and with outcome set  $X$  as  $\text{Tests}(A \rightarrow B, X)$ . The set of all tests of type  $A \rightarrow B$  will be denoted as

$$\text{Tests}(A \rightarrow B) := \bigcup_{X \in \text{Outcomes}} \text{Tests}(A \rightarrow B, X).$$

The sequential composition in  $\text{Transf}$  induces a sequential composition of tests, defined as

$$\{\mathcal{N}_y\}_{y \in Y} \circ \{\mathcal{M}_x\}_{x \in X} := \{\mathcal{N}_y \circ \mathcal{M}_x\}_{(x,y) \in X \times Y}. \quad (2)$$

By this definition, the sequential composition of a test in  $\text{Tests}(A \rightarrow B, X)$  with a test in  $\text{Tests}(B \rightarrow C, Y)$  yields a test in  $\text{Tests}(A \rightarrow C, X \times Y)$ . The *identity test on system A* is the test  $\{\mathcal{I}_A\}$  with singleton outcome set  $\{\varepsilon\}$ .

The parallel composition in  $\text{Transf}$  induces a parallel composition of tests, defined as

$$\{\mathcal{M}_x\}_{x \in X} \otimes \{\mathcal{N}_y\}_{y \in Y} := \{\mathcal{M}_x \otimes \mathcal{N}_y\}_{(x,y) \in X \times Y}. \quad (3)$$

By this definition, the parallel composition of a test in  $\text{Tests}(A \rightarrow A', X)$  with a test in  $\text{Tests}(B \rightarrow B', Y)$  yields a test in  $\text{Tests}(A \otimes B \rightarrow A' \otimes B', X \times Y)$ .

It is immediate to verify that the collection of all tests forms a SMC, where the systems are the same systems of  $\text{Transf}$ , the morphisms are indexed lists of morphisms in  $\text{Transf}$ , and the operations of sequential and parallel composition are defined as in Eqs. (2) and (3). We denote this category by  $\text{Tests}$ .

Summing up, the operational structure of a theory is described by a triple

$$\text{Op} := (\text{Transf}, \text{Outcomes}, \text{Tests}),$$

which specifies which transformations are physically possible, which outcomes herald these transformations, and which tests can be performed.

## 2.2 Probabilistic structure

The probabilistic structure of a theory consists of a rule to assign probabilities to the outcomes that are generated by non-deterministic circuits. Precisely, the rule is given by a function  $\text{Prob}$  which maps the scalars  $\text{Transf}(\mathbf{I} \rightarrow \mathbf{I})$  into real numbers in the interval  $[0, 1] \subset \mathbb{R}$  and satisfies the following properties

1. *Consistency*: for every outcome space  $\mathbf{X} \in \text{Outcomes}$  and for every test  $\{s_x\}_{x \in \mathbf{X}} \in \text{Tests}(\mathbf{I} \rightarrow \mathbf{I}, \mathbf{X})$  the function  $\text{Prob} \circ s$  is a probability distribution, i. e.  $\text{Prob}(s_x) \geq 0$  and  $\sum_{x \in \mathbf{X}} \text{Prob}(s_x) = 1$ .
2. *Independence*:  $\text{Prob}(s \otimes t) = \text{Prob}(s) \text{Prob}(t)$  for every pair of scalars  $s, t \in \text{Transf}(\mathbf{I} \rightarrow \mathbf{I})$ .

Here the consistency property guarantees that we can interpret  $\text{Prob}(s_x)$  as the probability of the outcome  $x \in \mathbf{X}$ . The independence property guarantees that experiments that involve only independent tests on two systems give rise to uncorrelated outcomes. As observed by Hardy [29, 28], independence is equivalent to the requirement that one can assign probabilities to a closed circuit in a way that is independent of the context. Note that the map  $\text{Prob}$  does not need to be onto: for example, in a deterministic theory the range of the map  $\text{Prob}$  are only the values 0 and 1.

The probabilistic structure endows the scalars in  $\text{Transf}$  with a structure of *test space* [39, 38]. Mathematically, a test space is a hypergraph equipped with function that assigns probabilities to vertices, under the condition that the sum of the probabilities is equal to 1 for every hyperedge. Note that the requirement of independence in the definition above adds extra structure to the test space, forcing a homomorphism between the monoid of scalars and the monoid of probabilities. Test spaces with additional structure have been recently considered in the characterization of contextuality [10, 11] and non-locality [3].

Combining the ingredients given so far, we can give the formal definition of an operational-probabilistic theory as a couple  $\Theta := (\text{Op}, \text{Prob})$ , or equivalently, as a quadruple  $\Theta = (\text{Transf}, \text{Outcomes}, \text{Tests}, \text{Prob})$ .

## 2.3 Quotient theories

The interaction between the operational structure and the probabilistic structure has major consequences. Eventually, it allows one to represent the category of physical transformations as a category of positive maps on ordered vector spaces. The steps that lead to this representation are summarized in the following.

### 2.3.1 The operation of quotient

The probabilistic structure brings with itself a natural notion of equivalent transformations. To spell out this notion clearly, it is convenient to introduce some notation: a transformation  $\rho$  of type  $\mathbf{I} \rightarrow \mathbf{A}$  will be called a *state of system A* and will be represented as

$$\boxed{\rho}^{\mathbf{A}} := \mathbf{I} \boxed{\rho}^{\mathbf{A}} \quad . \quad (4)$$

The set of all states of system  $\mathbf{A}$  will be denoted as  $\text{St}(\mathbf{A})$ . A transformation  $a$  of type  $\mathbf{A} \rightarrow \mathbf{I}$  will be called an *effect on system A* and will be represented as

$$\mathbf{A} \boxed{a} := \mathbf{A} \boxed{a}^{\mathbf{I}} \quad . \quad (5)$$

The set of all effects on system  $\mathbf{A}$  will be denoted as  $\text{Eff}(\mathbf{A})$ . Two transformations of type  $\mathbf{A} \rightarrow \mathbf{B}$ , say  $\mathcal{M}$  and  $\mathcal{M}'$ , are called *equivalent* iff

$$\text{Prob} \left( \left( \boxed{\rho}^{\mathbf{A}} \xrightarrow{\mathbf{A}} \boxed{\mathcal{M}}^{\mathbf{B}} \xrightarrow{\mathbf{B}} \boxed{E}^{\mathbf{I}} \right) \right) = \text{Prob} \left( \left( \boxed{\rho}^{\mathbf{A}} \xrightarrow{\mathbf{A}} \boxed{\mathcal{M}'}^{\mathbf{B}} \xrightarrow{\mathbf{B}} \boxed{E}^{\mathbf{I}} \right) \right) \quad (6)$$

for every system  $R \in \text{Sys}$ , every state  $\rho \in \text{St}(A \otimes R)$ , and every effect  $E \in \text{Eff}(B \otimes R)$ .

Note that the use of the “reference system”  $R$  is essential, unless the theory enjoys a property known as “local tomography”. [27, 8, 24]. When local tomography is not satisfied, checking the validity of Eq. (6) only for  $R = I$  may not be sufficient to guarantee its validity for arbitrary  $R$ . There are only two cases where the system  $R$  is not needed by default: the case of states (transformations of type  $I \rightarrow A$ ) and the case of effects (transformations of type  $A \rightarrow I$ ). For example, for two states  $\beta, \beta' \in \text{St}(B)$  Eq. (6) reads

$$\text{Prob} \left( \begin{array}{c} \boxed{\beta} \xrightarrow{B} \boxed{E} \\ \boxed{\rho} \xrightarrow{R} \end{array} \right) = \text{Prob} \left( \begin{array}{c} \boxed{\beta} \xrightarrow{B} \boxed{E} \\ \boxed{\rho} \xrightarrow{R} \end{array} \right) \quad \forall \rho \in \text{St}(R), \forall E \in \text{St}(B \otimes R),$$

which is equivalent to the condition

$$\text{Prob} \left( \begin{array}{c} \boxed{\beta} \xrightarrow{B} \boxed{b} \end{array} \right) = \text{Prob} \left( \begin{array}{c} \boxed{\beta} \xrightarrow{B} \boxed{b} \end{array} \right) \quad \forall b \in \text{St}(B).$$

We denote by  $[\mathcal{M}]$  the equivalence class of the transformation  $\mathcal{M}$  and by  $[\text{Transf}(A \rightarrow B)]$  the set of all equivalence classes of transformations of type  $A \rightarrow B$ . It is easy to see that the equivalence classes of transformations form an SMC, denoted by  $[\text{Transf}]$ , and that the operation of quotient is a forgetful (strong symmetric monoidal) functor from  $\text{Transf}$  to  $[\text{Transf}]$ . Furthermore, the equivalence classes of transformations induce equivalence classes of tests, defined as

$$[\{\mathcal{M}\}_{x \in X}] := \{[\mathcal{M}_x]\}_{x \in X}.$$

Also in this case the operation of quotient is a forgetful (strong symmetric monoidal) functor, from the SMC  $\text{Tests}$  to the SMC  $[\text{Tests}]$  consisting of equivalence classes of tests. In summary the operational structure  $\text{Op} = (\text{Transf}, \text{Outcomes}, \text{Tests})$  can be mapped functorially into a new operational structure

$$[\text{Op}] := ([\text{Transf}], \text{Outcomes}, [\text{Tests}]).$$

Defining the probabilistic structure  $[\text{Prob}]$  by the relation

$$[\text{Prob}](\llbracket s \rrbracket) := \text{Prob}(s) \quad \forall s \in \text{Transf}(I \rightarrow I),$$

it is easy to check that the couple  $[\Theta] = ([\text{Op}], [\text{Prob}])$  is an operational-probabilistic theory, which we call the *quotient theory*.

### 2.3.2 Axiomatic characterization of quotient theories

From now on, we will *only* consider quotient theories: given an operational-probabilistic theory  $\Theta = (\text{Transf}, \text{Outcomes}, \text{Tests}, \text{Prob})$ , we will assume that the operation of quotient has been already made. This is equivalent to requiring the following *separation axiom*:

**Axiom 1 (Separation)** *If  $\mathcal{M}$  and  $\mathcal{M}'$  are such that*

$$\text{Prob} \left( \begin{array}{c} \boxed{\rho} \xrightarrow{A} \boxed{\mathcal{M}} \xrightarrow{B} \boxed{E} \\ \boxed{\rho} \xrightarrow{R} \end{array} \right) = \text{Prob} \left( \begin{array}{c} \boxed{\rho} \xrightarrow{A} \boxed{\mathcal{M}'} \xrightarrow{B} \boxed{E} \\ \boxed{\rho} \xrightarrow{R} \end{array} \right) \quad \begin{array}{l} \forall R \in \text{Sys} \\ \forall \rho \in \text{St}(A \otimes R) \\ \forall E \in \text{Eff}(B \otimes R) \end{array}$$

*then  $\mathcal{M} = \mathcal{M}'$ .*

### 2.3.3 Embedding into ordered vector spaces

Quotient theories have one major feature, highlighted by the following

**Theorem 1** *Let  $\text{Transf}$  be the category of transformations in a quotient theory. Then, there exists a symmetric monoidal functor  $F$  from  $\text{Transf}$  to an SMC where the objects are partially order real vector spaces and the morphisms are positive linear maps. The functor  $F$  has the following properties:*

1.  $F(\mathbf{I}) = \mathbb{R}$
2. For every system  $A \in |\text{Transf}|$ , one has

$$\text{Span}_{\mathbb{R}} \{F(\alpha), \alpha \in \text{St}(A)\} \simeq F(A), \quad (7)$$

$$\text{Span}_{\mathbb{R}} \{F(a), a \in \text{Eff}(A)\} \simeq F(A)^*, \quad (8)$$

$$F(a) \circ F(\alpha) = \text{Prob}(a \circ \alpha) \quad \forall \alpha \in \text{St}(A), \forall a \in \text{Eff}(A). \quad (9)$$

3. For every pair of systems  $A, B \in |\text{Transf}|$  and every pair of transformations  $\mathcal{M}, \mathcal{M}' \in \text{Transf}(A \rightarrow B)$ , one has the separation property

$$F(\mathcal{M}' \otimes \mathcal{I}_R) = F(\mathcal{M} \otimes \mathcal{I}_R) \quad \forall R \in |\text{Transf}| \implies \mathcal{M}' = \mathcal{M}. \quad (10)$$

The correspondence carries over to the tests in  $\text{Tests}$ , which can be mapped functorially into an SMC of partially ordered vector spaces and indexed lists of positive maps.

An important point needs to be made here: in general the functor  $F$  is *neither faithful nor strong*. It is not faithful, because if the local tomography property does not hold, one can have  $F(\mathcal{M}') = F(\mathcal{M})$  even if  $\mathcal{M}' \neq \mathcal{M}$ . This is not in contradiction with the separation property of Eq. (10), which only implies that there exists some system  $R \in |\text{Transf}|$  such that  $F(\mathcal{M}' \otimes \mathcal{I}_R) \neq F(\mathcal{M} \otimes \mathcal{I}_R)$ . The separation property also implies that, if  $F$  is not faithful, it cannot be strong, i. e. one has

$$F(\mathcal{M} \otimes \mathcal{N}) \not\simeq F(\mathcal{M}) \otimes F(\mathcal{N}). \quad (11)$$

To some extent, this is a bug that needs to be fixed, because it would be very convenient to identify physical transformations with positive maps. The fix is provided by the separation property of Eq. (10), which guarantees that the transformation  $\mathcal{M}$  is in one-to-one correspondence with the indexed set  $\{F(\mathcal{M} \otimes \mathcal{I}_R) \mid R \in |\text{Transf}|\}$ . Using this fact, we can identify the transformation  $\mathcal{M}$  with the linear map

$$G(\mathcal{M}) = \bigoplus_{R \in |\text{Transf}|} F(\mathcal{M} \otimes \mathcal{I}_R), \quad (12)$$

which transforms elements of the vector space

$$G(A) := \bigoplus_{R \in |\text{Transf}|} F(A \otimes R) \quad (13)$$

into elements of the vector space

$$G(B) := \bigoplus_{R \in |\text{Transf}|} F(B \otimes R). \quad (14)$$

Endowing the vector spaces  $G(A)$  and  $G(B)$  with the direct sum cone inherited from the vector spaces  $F(A \otimes \mathbb{R})$  and  $F(B \otimes \mathbb{R})$ , respectively, one has that  $G(\mathcal{M})$  is a positive map.

Note that for two transformations  $\mathcal{M} \in \text{Transf}(A \rightarrow B)$  and  $\mathcal{N} \in \text{Transf}(B \rightarrow C)$  one has

$$G(\mathcal{N}) \circ G(\mathcal{M}) = \bigoplus_{R \in |\text{Transf}|} F[(\mathcal{N} \circ \mathcal{M}) \otimes \mathcal{I}_R] \equiv G(\mathcal{N} \circ \mathcal{M}), \quad (15)$$

which shows that  $G$  is a functor. By definition 12, the functor  $G$  is faithful.

In addition, for two transformations  $\mathcal{A} \in \text{Transf}(A \rightarrow A')$  and  $\mathcal{B} \in \text{Transf}(B \rightarrow B')$  one has

$$G(\mathcal{A}) \otimes G(\mathcal{B}) \simeq \bigoplus_{R \in |\text{Transf}|} F(\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{I}_R) \equiv G(\mathcal{A} \otimes \mathcal{B}), \quad (16)$$

up to natural isomorphism. In other words,  $G$  is a strong monoidal functor. Summarizing, we obtained the following

**Theorem 2** *Let  $\text{Transf}$  be the category of transformations in a quotient theory. Then, there exists a faithful strong symmetric monoidal functor  $G$  between  $\text{Transf}$  and an SMC of partially order real vector spaces and positive linear maps, defined as in Eq. (12). The functor  $G$  is such that, for every system  $A$ , the following conditions are satisfied:*

$$\text{Span}_{\mathbb{R}} \{G(\alpha) \mid \alpha \in \text{St}(A)\} \simeq F(A), \quad (17)$$

$$\text{Span}_{\mathbb{R}} \{G(a) \mid a \in \text{Eff}(A)\} \simeq F(A)^*, \quad (18)$$

and

$$G(a \circ \rho) = \text{Prob}(a \circ \rho) G(1) \quad \forall \alpha \in \text{St}(A), \forall a \in \text{Eff}(A). \quad (19)$$

In other words, the category  $\text{Transf}$  can be *identified* with an SMC of ordered vector spaces and positive maps. This fact is very useful because it allows one to define linear combinations of transformations, such as

$$\mathcal{X} = \sum_x x_i \mathcal{M}_i \quad \{x_i\} \subset \mathbb{R}, \{\mathcal{M}_i\} \subset \text{Transf}(A \rightarrow B). \quad (20)$$

The real vector space spanned by  $\text{Transf}(A \rightarrow B)$  will be denoted  $\text{Transf}_{\mathbb{R}}(A \rightarrow B)$ . Note that, typically,  $\text{Transf}_{\mathbb{R}}(A \rightarrow B)$  is a proper subspace of  $L[G(A), G(B)]$ , the space of all linear maps from the vector space  $G(A)$  to the vector space  $G(B)$ . For example, one has  $\text{Transf}_{\mathbb{R}}(I \rightarrow I) \simeq \mathbb{R}$  whereas  $L[G(I), G(I)]$  is generally an infinite-dimensional vector space. In the case of states and effects, we use the special notations

$$\text{St}_{\mathbb{R}}(A) := \text{Transf}_{\mathbb{R}}(I \rightarrow A)$$

and

$$\text{Eff}_{\mathbb{R}}(A) := \text{Transf}_{\mathbb{R}}(A \rightarrow I).$$

We say that a system is *finite-dimensional* iff the dimension of  $\text{St}_{\mathbb{R}}(A)$  [or, equivalently, the dimension of  $\text{Eff}_{\mathbb{R}}(A)$ ] is finite.

## 2.4 Physicalizing the readout

In the basic framework we introduced outcome spaces as abstract index sets labelling different possible processes. At the intuitive level, however, the outcome of a test is written on some physical system and can be read out from it. In order to express this fact one can require that every test arises from a transformation followed by a measurement on one system. This requirement is expressed by the following

**Axiom 2 (Physicalization of readout)** *For every pair of systems  $A, B \in |\text{Transf}|$ , every outcome space  $X \in \text{Outcomes}$ , and every test  $\{\mathcal{M}_x\}_{x \in X} \in \text{Tests}(A \rightarrow B, X)$  there exist a system  $C \in |\text{Transf}|$ , a transformation  $\mathcal{M} \in \text{Transf}(A \rightarrow B \otimes C)$ , and a test  $\{c_x\}_{x \in X} \in \text{Tests}(C \rightarrow I, X)$  such that*

$$\boxed{A} \boxed{\mathcal{M}_x} \boxed{B} = \boxed{A} \boxed{\mathcal{M}} \boxed{B} \boxed{C} \boxed{c_x} \quad \forall x \in X. \quad (21)$$

In principle, one could also require that the test  $\{c_x\}_{x \in X} \in \text{Tests}(C \rightarrow I, X)$  can distinguish perfectly among a set of states, as it was done in Ref. [12]. Similarly, one could set up more requirements on the transformation  $\mathcal{M}$ , e. g. requiring it to be of the form  $\mathcal{M} = \sum_{x \in X} \mathcal{M}_x \otimes \gamma_x$ , where  $\{\gamma_x\}_{x \in X}$  is a set of perfectly distinguishable states. However, these extra requirements are less essential than the basic assumption that the theory should be able to model the readout process, as in Eq. (21).

## 3 Pure states, pure transformations, and reversible transformations

The framework of operational-probabilistic theories allows one to define the familiar notion of pure state: a state  $\alpha \in \text{St}(A)$  is *pure* iff for every set of states  $\{\alpha_i\}_{i=1}^N \subset \text{St}(A)$  one has the implication

$$\sum_{i=1}^N \alpha_i = \alpha \implies \alpha_i = p_i \alpha, \quad \forall i \in \{1, \dots, N\}, \quad (22)$$

for some probabilities  $\{p_i\}$ . We denote the set of pure states of system  $A$  as  $\text{PurSt}(A)$ .

A similar notion can be put forward for transformations: a transformation  $\mathcal{M} \in \text{Transf}(A \rightarrow B)$  is *pure* iff for every set of transformations  $\{\mathcal{M}_i\}_{i=1}^N \subset \text{Transf}(A \rightarrow B)$  one has the implication

$$\sum_{i=1}^N \mathcal{M}_i = \mathcal{M} \implies \mathcal{M}_i = p_i \mathcal{M}, \quad \forall i \in \{1, \dots, N\}, \quad (23)$$

for some probabilities  $\{p_i\}$ . We denote the set of pure transformations from  $A$  to  $B$  as  $\text{PurTransf}(A \rightarrow B)$ . For effects, we use the notation  $\text{PurEff}(A) := \text{PurTransf}(A \rightarrow I)$ .

Finally, another important notion is the notion of reversible transformation. This is a primitive notion, which does not even need the probabilistic structure: a reversible transformation is just an isomorphism in the category  $\text{Transf}$ . Explicitly, a transformation  $\mathcal{U} \in \text{Transf}(A \rightarrow B)$  is *reversible* iff there exists another transformation  $\mathcal{U}^{-1} \in \text{Transf}(B \rightarrow A)$  such that

$$\boxed{A} \boxed{\mathcal{U}} \boxed{B} \boxed{\mathcal{U}^{-1}} \boxed{A} = \boxed{A} \boxed{\mathcal{I}} \boxed{A} \quad \text{and} \quad \boxed{B} \boxed{\mathcal{U}^{-1}} \boxed{A} \boxed{\mathcal{U}} \boxed{B} = \boxed{B} \boxed{\mathcal{I}} \boxed{B}. \quad (24)$$

The set of reversible transformations of type  $A \rightarrow B$  will be denoted by  $\text{RevTransf}(A \rightarrow B)$ . The notion of pure state, pure transformation, and reversible transformation will be used in section 5 for the formulation of the Purification Principle.



## 4 Causality

Once the basic framework of operational-probabilistic theories has been defined, one can formulate axioms that define classes of theories sharing common features. A particularly basic axiom is causality, which reads as follows

**Axiom 3 (Causality)** *For every system  $A \in |\text{Transf}|$  there exists an effect  $\text{Tr}_A \in \text{Eff}(A)$ , called the trace (or the deterministic effect [12, 13]) and graphically represented as  $\overline{B} \boxed{\text{Tr}}$ , such that*

$$\sum_{x \in X} a_x = \text{Tr}_A \quad \forall \{a_x\}_{x \in X} \in \text{Tests}(A \rightarrow I, X), \quad \forall X \in \text{Outcomes}. \quad (25)$$

From the definition and from the fact that  $\text{Tests}$  is an SMC, it follows immediately that the trace satisfies the following relations

$$\begin{aligned} \text{Tr}_{A \otimes B} &= \text{Tr}_A \otimes \text{Tr}_B \\ \text{Tr}_I &= 1. \end{aligned} \quad (26)$$

### 4.1 Deterministic transformations

We call a transformation  $\mathcal{M} \in \text{Transf}(A \rightarrow B)$  *deterministic*<sup>2</sup> iff

$$\overline{A} \boxed{\mathcal{M}} \overline{B} \boxed{\text{Tr}} = \overline{A} \boxed{\text{Tr}}. \quad (27)$$

Combining the definition with Eqs. (26) one obtains that the deterministic transformations form a symmetric monoidal subcategory of  $\text{Transf}$ , which we denote by  $\text{DetTransf}$ . An equivalent way to state the causality axiom, put forward by Coecke and Lal in Refs. [21, 20], is to start from a distinguished SMC  $\text{DetTransf} \hookrightarrow \text{Transf}$  and to impose that the tensor unit  $I$  is terminal in  $\text{DetTransf}$ .

### 4.2 Marginal states and marginal transformations

The trace allows one to introduce a canonical notion of marginal state: given a state  $\sigma \in \text{St}(A \otimes B)$ , the marginal of  $\sigma$  on system  $A$  is the state  $\rho \in \text{St}(A)$  defined by

$$\overline{\rho} \boxed{A} := \left( \overline{\sigma} \begin{array}{c} \overline{A} \\ \overline{B} \end{array} \boxed{\text{Tr}} \right).$$

Conversely, we say that  $\sigma$  is an *extension of  $\rho$  to system  $A \otimes B$* .

The same definition can be phrased for general transformations: the *marginal of a transformation  $\mathcal{N} \in \text{Transf}(A \rightarrow B \otimes C)$  on system  $B$*  is the transformation  $\mathcal{M} \in \text{Transf}(A \rightarrow B)$  defined as

$$\overline{A} \boxed{\mathcal{M}} \overline{B} := \left( \overline{A} \boxed{\mathcal{N}} \begin{array}{c} \overline{B} \\ \overline{C} \end{array} \boxed{\text{Tr}} \right). \quad (28)$$

When this is the case, we say that  $\mathcal{N}$  is an *extension of  $\mathcal{M}$  to system  $B \otimes C$* .

<sup>2</sup>In the original works [12, 13] the deterministic transformations were defined in a different way, starting from tests with singleton outcome set. However, in a quotient theory where the causality axiom holds, the definition of Refs. [12, 13] is equivalent to the one given here.

## 5 Purification

Purification is the requirement that every mixed state can be modelled as the marginal of a pure state in an essentially unique way. For finite-dimensional systems, the axiom reads:

**Axiom 4 (Purification)** *For every system  $A \in |\text{Transf}|$  and for every state  $\rho \in \text{St}(A)$  there exists a system  $B$ , called the purifying system, and a pure state  $\Psi \in \text{PurSt}(A \otimes B)$  such that*

$$\left( \Psi \right)_{\substack{A \\ B}}^{\text{Tr}} = \rho^A.$$

Moreover, for every system  $B \in |\text{Transf}|$  and for every pair of pure states  $\Psi, \Psi' \in \text{PurSt}(A \otimes B)$  one has the implication

$$\left( \Psi' \right)_{\substack{A \\ B}}^{\text{Tr}} = \left( \Psi \right)_{\substack{A \\ B}}^{\text{Tr}} \implies \left( \Psi' \right)_{\substack{A \\ B}}^{\text{Tr}} = \left( \Psi \right)_{\substack{A \\ B}}^{\text{Tr}} \circ \mathcal{U}_B$$

for a reversible transformation  $\mathcal{U} \in \text{RevTransf}(B)$ .

The Purification Axiom is the operational translation of (some features of) the GNS construction [26, 34] in the special case of finite-dimensional irreducible matrix algebras.

### 5.1 Consequences of purification

Given the importance of the GNS construction it is probably not surprising that the Purification Axiom has a large number of consequences, some of which are listed in the following.

#### 5.1.1 Transitivity of reversible transformations on pure states

An immediate consequence of purification is that every two pure states of a system are connected by a reversible transformation:

**Proposition 1** *Suppose that the theory  $\Theta$  satisfies purification. Then, for every system  $A \in |\text{Transf}|$  and every pair of pure states  $\alpha, \alpha' \in \text{PurSt}(A)$  there exists a reversible transformation  $\mathcal{U} \in \text{RevTransf}(A)$  such that  $\alpha' = \mathcal{U} \circ \alpha$ .*

This property had been used as an axiom in Hardy's 2001 axiomatization [27] and in subsequent axiomatizations [23, 30].

#### 5.1.2 Steering

Another consequence of purification is the fact that every decomposition of a mixed state can be induced by a measurement on the purifying system:

**Proposition 2** *Suppose that the theory  $\Theta$  satisfies purification. Let  $\rho \in \text{St}(A)$  be a state of a generic system  $A$  and let  $\Psi \in \text{PurSt}(A \otimes B)$  be a purification of  $\rho$ . Then, for every outcome space  $X \in \text{Outcomes}$  and every test  $\{\rho_x\}_{x \in X} \in \text{Tests}(A \rightarrow B, X)$  such that  $\sum_{x \in X} \rho_x = \rho$  there exists a measurement  $\{b_x\}_{x \in X} \in \text{Tests}(B \rightarrow I, X)$  such that*

$$\left( \rho_x \right)^A = \left( \Psi \right)_{\substack{A \\ B}}^{b_x} \quad \forall x \in X.$$

## 6 Faithfulness

Until now we did not assume the existence of mixed states: the general framework of operational-probabilistic theories describes also theories that have no mixed states at all, such as deterministic classical computation. As a matter of fact, even the purification axiom is satisfied by deterministic classical computation—in a trivial way, since there is no mixed state to be purified there.

We now require that in our theory there exist mixed states and, in particular, that there exist states that are “sufficiently mixed”, according to the following definition: we call a state  $\omega \in \text{St}(A)$  *faithful* iff for every system  $B$  and for every pair of transformations  $\mathcal{M}, \mathcal{M}' \in \text{Transf}(A \rightarrow B)$  the condition

$$\left[ \begin{array}{c} \text{A} \\ \boxed{\mathcal{M}} \\ \text{B} \\ \hline \sigma \\ \text{C} \end{array} \right] = \left[ \begin{array}{c} \text{A} \\ \boxed{\mathcal{M}'} \\ \text{B} \\ \hline \sigma \\ \text{C} \end{array} \right] \quad \forall C \in |\text{Transf}|, \forall \sigma \in \text{St}(A \otimes C) : \sigma \text{ is an extension of } \omega \quad (29)$$

implies  $\mathcal{M} = \mathcal{M}'$ . The definition of faithful states given here coincides with the  $C^*$ -algebraic definition faithful states in the finite dimensional setting. Faithful states exist in every convex theory where the set of states  $\text{St}(A)$  is convex and finite-dimensional [12, 13]: in that case, they are nothing but the states in the interior of the state space. Rather than requiring convexity, here we just require the existence of a faithful state for every system:

**Axiom 5 (Faithfulness)** *For every system  $A \in |\text{Transf}|$ , there exists at least one faithful state  $\omega_A \in \text{St}(A)$ .*

## 7 Consequences of purification and faithfulness

The full power of purification appears in conjunction with the faithfulness axiom. The core result is the existence of a pure state that establishes an injective correspondence between transformations and states.

### 7.1 The state-transformation isomorphism

Once the definitions are in place, it is easy to establish a general isomorphism that has the operational features of the Choi isomorphism for finite-dimensional matrix algebras [16]:

**Proposition 3** *Suppose that the theory  $\Theta$  satisfies Axioms 2, 4, and 5. Then, for every system  $A \in |\text{Transf}|$  and every purification of the faithful state of  $A$ , say  $\Psi \in \text{PurSt}(A \otimes C)$ , has the following property:*

$$\left[ \begin{array}{c} \text{A} \\ \boxed{\mathcal{M}} \\ \text{B} \\ \hline \Phi \\ \text{C} \end{array} \right] = \left[ \begin{array}{c} \text{A} \\ \boxed{\mathcal{M}'} \\ \text{B} \\ \hline \Phi \\ \text{C} \end{array} \right] \implies \mathcal{M} = \mathcal{M}' \quad \forall B \in |\text{Transf}|, \forall \mathcal{M}, \mathcal{M}' \in \text{Transf}(A \rightarrow B). \quad (30)$$

We call the correspondence  $\mathcal{M} \mapsto \Phi_{\mathcal{M}} := (\mathcal{M} \otimes \mathcal{I}_C) \circ \Phi$  the *state-transformation isomorphism* [12, 13]. The fact that the state-transformation isomorphism is set up by a *pure* state  $\Phi$  has major consequences, briefly sketched in the next subsections:

## 7.2 No information without disturbance

The most consequence of the pure state-transformation isomorphism is the No Information Without Disturbance principle, which states that every test  $\{\mathcal{M}_x\}_{x \in X} \in \text{Tests}(A \rightarrow A, X)$  satisfying the “no disturbance condition”

$$\sum_{x \in X} \mathcal{M}_x = \mathcal{I}_A$$

must also satisfy the “no information condition”

$$\mathcal{M}_x = p_x \mathcal{I}_A \quad \forall x \in X$$

for some probability distribution  $\{p_x\}_{x \in X}$ .

## 7.3 Purification of transformations

We have seen that the purification of states expresses the operational content of the GNS construction for finite dimensional quantum systems. What about Stinespring’s theorem [37], whose statement includes the GNS construction as a special case? Interestingly, using the state-transformation isomorphism in combination with axioms 2 and 5 one can obtain the operational version of Stinespring’s theorem from the purification axiom:

**Proposition 4 (Purification of transformations)** *Suppose that the theory  $\Theta$  satisfies Axioms 2, 4, and 5. For every pair of systems  $A, B \in |\text{Transf}|$  and for every transformation  $\mathcal{M} \in \text{Transf}(A \rightarrow B)$  there exists a system  $C$  and a pure transformation  $\mathcal{P} \in \text{PurTransf}(A \rightarrow B \otimes C)$  such that*

$$\begin{array}{c} A \quad B \\ \boxed{\mathcal{P}} \\ C \quad \text{Tr} \end{array} = \begin{array}{c} A \quad B \\ \boxed{\mathcal{M}} \end{array} .$$

Moreover, for every system  $C \in |\text{Transf}|$  and for every pair of pure transformations  $\mathcal{P}, \mathcal{P}' \in \text{PurTransf}(A \otimes B)$  one has the implication

$$\begin{array}{c} A \quad B \\ \boxed{\mathcal{P}'} \\ C \quad \text{Tr} \end{array} = \begin{array}{c} A \quad A \\ \boxed{\mathcal{P}} \\ C \quad \text{Tr} \end{array} \implies \begin{array}{c} A \quad B \\ \boxed{\mathcal{P}'} \\ C \end{array} = \begin{array}{c} A \quad B \\ \boxed{\mathcal{P}} \\ C \quad \mathcal{U} \quad C \end{array}$$

for a reversible transformation  $\mathcal{U} \in \text{RevTransf}(C)$ .

This result concludes our review of dilation-type results that can derived axiomatically in the framework of operational-probabilistic theories. As the reader might have noticed, we did not discuss dilation theorems of the Naimark- [31] and Ozawa-type [32]. They are based on a different conceptual ingredient, namely the notion of *sharp measurement*. A discussion in this direction can be found in Ref.[15], which derived upper bounds on quantum nonlocality and contextuality from a Naimark-type dilation of measurements to sharp measurements.

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